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On the motion due to sources and sinks distributed along the vertical boundary of a rotating fluid

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The motion induced by sources and sinks distributed along the vertical side wall of a cylinder filled with fluid and rotating about a vertical axis is considered. Vertical motion, and hence vertical transport, is confined to a vertical boundary layer of thickness $E^{\frac{1}{3}}$, where E is the Ekman number. The horizontal transport occurs through the interior of the fluid. The Ekman layers do not play any active role in the transport process. The results of a very simple experiment confirm these conclusions.

1. Introduction

In the present paper we shall consider the flow induced by radially injecting and withdrawing fluid through the vertical side wall of a cylinder filled with viscous, incompressible fluid. The top and bottom walls of the cylinder are assumed to be rigid and the cylinder is rotating about a vertical axis with constant angular velocity. Without loss of generality, we can, for instance, consider that fluid is injected at one point located at a certain height and longitude and that it is withdrawn at another point located at a different height and longitude (see figure 1, plate 1). Because of the Taylor–Proudman theorem, we can anticipate that vertical motions will take place within a thin boundary layer girdling the cylinder, of the type first discussed by Stewartson (1957), and that therefore the vertical transport will be effected through this boundary layer. It is, however, not clear *a priori* how the horizontal transport will be achieved, i.e. whether it will also occur through this vertical boundary layer, or by means of motions in the interior of the fluid, or still by some devious way via the Ekman layers present at the top and bottom boundaries as in the case of the similar though not identical problems considered by Lewellen (1965) and Hide (1967). This question, which we propose to answer, is of some relevance in many geophysical problems stemming from models of atmospheric and oceanic currents driven by such source–sink distributions.

2. Formulation

The equations of motion for a viscous, incompressible fluid, written in a frame fixed with respect to the cylindrical container which rotates with angular velocity Ω about the vertical, are:

$$\mathbf{v}'.\nabla'\mathbf{v}'+2\Omega\mathbf{k}\times\mathbf{v}'+\rho^{-1}\nabla'P'-\tfrac{1}{2}\Omega^2\nabla'|\mathbf{k}\times\mathbf{r}'|^2-\nu\nabla'^2\mathbf{v}'=0, \tag{2.1}$$

$$\nabla'.\mathbf{v}'=0. \tag{2.2}$$

Denoting by Q the (volume) flux of fluid entering the side wall and by H the height of the cylinder, we introduce dimensionless variables as follows:

$$\mathbf{r}' = H\mathbf{r}, \quad \mathbf{v}' = \frac{Q}{H^2}\mathbf{v}, \quad P' = \tfrac{1}{2}\Omega^2\rho H^2|\mathbf{k}\times\mathbf{r}|^2 + \rho\frac{\Omega Q}{H}p. \tag{2.3}$$

Substituting these expressions in (2.1), (2.2) we can write

$$\epsilon(\mathbf{v}.\nabla)\,\mathbf{v} + 2\mathbf{k}\times\mathbf{v} = -\nabla p + E\nabla^2\mathbf{v}, \tag{2.4}$$

$$\nabla.\mathbf{v} = 0, \tag{2.5}$$

where

$$\epsilon = Q/\Omega H^3 \tag{2.6}$$

and

$$E = \nu/\Omega H^2 \tag{2.7}$$

are respectively the Rossby and Ekman numbers. We shall restrict our analysis to the case of small ϵ (i.e. small mass flux) and neglect the non-linear terms. The boundary-value problem which we propose to solve is therefore

$$\left.\begin{aligned} 2\mathbf{k}\times\mathbf{v} &= -\nabla p + E\nabla^2\mathbf{v}, \\ \nabla.\mathbf{v} &= 0, \\ \mathbf{v} &= I(z,\theta)\mathbf{n} \quad \text{on} \quad r = R, \\ \mathbf{v} &= 0 \quad \text{on} \quad z = 0, 1, \end{aligned}\right\} \tag{2.8}$$

where $\mathbf{n}$ is the outer normal, R the dimensionless radius of the cylinder and $I(z,\theta)$ is an arbitrary function of z and θ which specifies the injection velocity and which is such that no net mass flux enters the cylinder, i.e.

$$\int_0^{2\pi} d\theta \int_0^1 I(z,\theta)\,dz = 0. \tag{2.9}$$

In the remainder of the paper, we shall never have to rely explicitly on the fact that the cross-section of the cylinder is circular. Actually, the choice of this cross-section is made purely for the sake of the simplicity since the procedure to be followed is valid for a cylinder of arbitrary cross-section, the only difference residing in the use of the 'natural' co-ordinates of the surface instead of the cylindrical ones.

3. Solution

The actual solution of the boundary-value problem (2.8) is quite straightforward but will contain lengthy expressions for various representations of the velocity field valid in different regions of the fluid. To prevent some of the arguments from being obscured by the inevitable algebra we shall first outline the basic steps to be followed, stating the results which we shall subsequently prove.

Using boundary-layer concepts, three distinct regions will emerge from the analysis: the top and bottom horizontal Ekman layers of thickness $E^{\frac{1}{2}}$, the side wall boundary layer made up by an inner layer of thickness $E^{\frac{1}{3}}$ and by an outer layer of thickness $E^{\frac{1}{4}}$, and finally the interior region. Using a procedure by now standard, we shall only consider the flow outside the Ekman layers which can be adequately accounted for by appropriate boundary conditions on the vertical velocity of the overlying or underlying field. Mathematically, this will enable us to reduce the order of the z-differentiation and in particular to treat the Laplacian occurring in (2.8) as if it were the two-dimensional horizontal one. These modified boundary conditions will also enable us to show that the interior flow is irrotational. We shall then turn to an investigation of the flow in the vicinity of the side wall. Denoting by a tilde the $E^{\frac{1}{3}}$ layer fields, by a bar the $E^{\frac{1}{4}}$ layer fields, by capitals the interior fields, and writing the E-dependence of these fields explicitly, we find that the side wall boundary conditions can be written as:

$$E^{a+\frac{1}{4}}\overline{u}+E^{b+\frac{1}{3}}\tilde{u}+E^{c}U = \tilde{I}(z,\theta)+\mathscr{I}(\theta), \tag{3.1}$$

$$E^{a}\overline{v}+E^{b}\tilde{v}+E^{c}V = 0, \tag{3.2}$$

$$E^{a+\frac{1}{4}}\overline{w}+E^{b}\tilde{w} = 0, \tag{3.3}$$

where $\overline{u}$, $\tilde{u}$, U, etc. ... are $O(1)$. $\mathscr{I}(\theta)$ is defined as the z-average of I, viz.

$$\mathscr{I}(\theta) = \int_0^1 I(z,\theta)\,dz \tag{3.4}$$

and
$$\tilde{I}(z,\theta) = I(z,\theta)-\mathscr{I}(\theta). \tag{3.5}$$

In order to deduce the values of a, b, c (i.e. the magnitude of the various fields) we shall make use of the following three properties: (i) the interior fields are independent of z; (ii) $\overline{u}$ and $\overline{v}$ are also independent of z, while $\overline{w}$ has a linear z-dependence; (iii) only $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ have a large degree of functional arbitrariness in their z-dependence, although the z-average of $\tilde{u}$ and $\tilde{v}$ is zero. Once these properties have been derived, we shall immediately be able to deduce that $b = -\frac{1}{3}$ and that the boundary conditions for the $E^{\frac{1}{3}}$ layer fields are

$$\tilde{u} = \tilde{I}(z,\theta), \quad \tilde{v} = 0, \quad \tilde{w} = 0. \tag{3.6}$$

We shall also show that $a = 0$ and $c = 0$, i.e. that

$$U(R,\theta) = \mathscr{I}(\theta) \tag{3.7}$$

and
$$\overline{v} = -V(R,\theta). \tag{3.8}$$

Using the appropriate equations together with the boundary conditions (3.6), (3.7) and (3.8) we shall then solve three boundary-value problems for the $E^{\frac{1}{3}}$ layer, the interior and the $E^{\frac{1}{4}}$ layer respectively.

Let us now proceed with the actual solution. Taking advantage of the smallness of E we expand the pressure and velocity in powers of $E^{\frac{1}{12}}$. The choice of this exponent is suggested by the results obtained by Robinson (1959) and Greenspan & Howard (1963), who considered Stewartson-type vertical shear layers girdling rigid walls. Formally, we can therefore write

$$\mathbf{v} = E^{c}\sum_{n=0} E^{\frac{1}{12}n}\mathbf{V}_n. \tag{3.9}$$

The factor E^c, where c is an unknown constant, is introduced to allow for the possibility that the first non-zero term is not of order one. Substituting in (2.8), we deduce the interior equations, viz.

$$2\mathbf{k}\times\mathbf{V}_0 = -\nabla P_0, \tag{3.10}$$

$$\nabla.\mathbf{V}_0 = 0. \tag{3.11}$$

Note that $\mathbf{V}_i$, P_i satisfy the same set of equations for $i = 1, 2, \ldots, 11$. Taking the curl of (3.10) we get the Taylor–Proudman theorem, namely

$$\partial \mathbf{V}_0 / \partial z = 0, \tag{3.12}$$

which states that the interior velocity is independent of z (property (i)). In addition to the Taylor–Proudman theorem, (3.10) shows that the horizontal velocity is geostrophic (i.e. isobars coincide with streamlines). The fact that the pressure can be considered as a stream function for the horizontal velocity can be seen by taking the cross-product of (3.10) by $\mathbf{k}$:

$$\mathbf{V}_0 - (\mathbf{k}.\mathbf{V}_0)\mathbf{k} = -\tfrac{1}{2}\nabla \times (\mathbf{k}P_0). \tag{3.13}$$

We shall make use of this well-known result later on.

In order to satisfy the inviscid and no-slip boundary conditions at the top and bottom boundaries, we must add boundary-layer corrections to the interior velocity $\mathbf{V}_0$. The equations governing these boundary-layer corrections are the usual Ekman boundary-layer ones, which, because of their simplicity, can be completely solved without a knowledge of the underlying or overlying fields. This very convenient property has been used to derive appropriate boundary conditions for the overlying or underlying fields, viz.

$$\left.\begin{aligned} w &= -\tfrac{1}{2}E^{\frac{1}{2}}(w_z - \zeta) \quad \text{at} \quad z = 0, \\ w &= +\tfrac{1}{2}E^{\frac{1}{2}}(w_z - \zeta) \quad \text{at} \quad z = 1, \end{aligned}\right\} \tag{3.14}$$

where $\zeta = \mathbf{k}.\nabla \times \mathbf{v}$ is the vertical component of the vorticity. We shall not reproduce the derivation of (3.14) which can be found in Charney & Eliassen (1949), Greenspan & Howard (1963), Barcilon (1964), Jacobs (1964), among others. An improved approximation which would be necessary if we were to carry the formal expansion in $E^{\frac{1}{12}}$ beyond $O(E^{\frac{1}{2}})$ is given below:

$$\left.\begin{aligned} w &= \tfrac{1}{2}E^{\frac{1}{2}}(w_z - \zeta) + \tfrac{1}{16}E^{\frac{3}{2}}\nabla^2(w_z - 3\zeta) + O(E^2) \quad \text{at} \quad z = 1, \\ w &= -\tfrac{1}{2}E^{\frac{1}{2}}(w_z - \zeta) - \tfrac{1}{16}E^{\frac{3}{2}}\nabla^2(w_z - 3\zeta) + O(E^2) \quad \text{at} \quad z = 0. \end{aligned}\right\} \tag{3.15}$$

It should be noted that these boundary conditions can be used not only for the interior fields, but also for the side wall boundary layer. Applying (3.14) to the interior field, we immediately deduce that

$$w_0 \equiv 0, \quad \nabla \times \mathbf{V}_0 = 0; \tag{3.16}$$

i.e. the interior flow is two-dimensional and irrotational. A repeated use of (3.15) would show that these features of the interior flow hold to much higher order than $o(E^{\frac{1}{12}})$. From (3.16) and (3.13) we deduce that

$$\nabla^2 P_0 = 0, \tag{3.17}$$

where ∇^2 is the two-dimensional horizontal Laplacian operator. To obtain a closed boundary-value problem for the interior, we must derive the appropriate boundary condition for P_0 at the vertical wall of the cylinder, which in turn requires that we consider the side wall boundary layer.

Vertical shear layers of this type were first considered by Stewartson (1957) for the case in which they are detached (or free), then by Robinson (1959) and

Greenspan & Howard (1963) for the case in which they are hugging a rigid wall. In all these investigations the flow was axially symmetric and these boundary layers had no azimuthal dependence. A free shear layer of this type which had an azimuthal dependence was first investigated by Jacobs (1964) in connexion with the Taylor column problem. In the present analysis, we are also concerned with an azimuthally dependent boundary layer, but since it hugs a rigid wall we cannot use Jacobs's results. Whether detached or not, the boundary-layer equations are identical, viz.

$$\left.\begin{aligned}\frac{\partial w}{\partial z} = -\tfrac{1}{2}E\frac{\partial^3 v}{\partial r^3}, \quad \frac{\partial v}{\partial z} = \tfrac{1}{2}E\frac{\partial^3 w}{\partial r^3},\\ \frac{\partial u}{\partial r}+\frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z} = 0.\end{aligned}\right\} \tag{3.18}$$

Let us first consider the $E^{\frac{1}{3}}$ part of this boundary layer. On introducing the stretched co-ordinate

$$\rho = (R-r)\,E^{-\frac{1}{3}}, \tag{3.19}$$

and omitting some simple calculations, we can see that the boundary-layer fields must be scaled thus:

$$\left.\begin{aligned}u &= E^{b+\frac{1}{3}}\tilde{u}(\rho,\theta,z),\\ v &= E^{b}\tilde{v}(\rho,\theta,z),\\ w &= E^{b}\tilde{w}(\rho,\theta,z),\end{aligned}\right\} \tag{3.20}$$

where $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are of $O(1)$ and satisfy the following equations:

$$\left.\begin{aligned}\tilde{v}_z = -\tfrac{1}{2}\tilde{w}_{\rho\rho\rho}, \quad \tilde{w}_z = \tfrac{1}{2}\tilde{v}_{\rho\rho\rho},\\ -\tilde{u}_\rho + R^{-1}\tilde{v}_\theta + \tilde{w}_z = 0.\end{aligned}\right\} \tag{3.21}$$

On substituting (3.20) in (3.14), we see that the boundary condition for $\tilde{w}$ is

$$\tilde{w} = 0 \quad \text{at} \quad z = 0, 1. \tag{3.22}$$

Let us therefore look for solutions of (3.21) of the form (property (iii)):

$$\tilde{u} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} e^{-\lambda_k\rho}\,[\tilde{u}_{nk}\cos n\theta + \tilde{u}'_{nk}\sin n\theta]\cos k\pi z, \tag{3.23}$$

$$\tilde{v} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} e^{-\lambda_k\rho}\,[\tilde{v}_{nk}\cos n\theta + \tilde{v}'_{nk}\sin n\theta]\cos k\pi z, \tag{3.24}$$

$$\tilde{w} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} e^{-\lambda_k\rho}\,[\tilde{w}_{nk}\cos n\theta + \tilde{w}'_{nk}\sin n\theta]\sin k\pi z. \tag{3.25}$$

Substituting (3.23)–(3.25) in (3.21) we get

$$\left.\begin{aligned}-k\pi\tilde{v}_{nk} = \tfrac{1}{2}\lambda_k^3\tilde{w}_{nk}, \quad k\pi\tilde{w}_{nk} = -\tfrac{1}{2}\lambda_k^3\tilde{v}_{nk},\\ (n/R)\,\tilde{v}_{nk}+k\pi\tilde{w}_{nk} = -\lambda_k\tilde{u}_{nk},\end{aligned}\right\} \tag{3.26}$$

and

$$\left.\begin{aligned}-k\pi\tilde{v}'_{nk} = \tfrac{1}{2}\lambda_k^3\tilde{w}'_{nk}, \quad k\pi\tilde{w}'_{nk} = -\tfrac{1}{2}\lambda_k^3\tilde{v}'_{nk},\\ -(n/R)\,\tilde{v}_{nk}+k\pi\tilde{w}'_{nk} = -\lambda_k\tilde{u}'_{nk}.\end{aligned}\right\} \tag{3.27}$$

Recalling that $Re\,(\lambda_k)$ must be positive, we can see that there are three roots for λ_k, viz.

$$(2k\pi)^{\frac{1}{3}}, \quad (2k\pi)^{\frac{1}{3}}(\tfrac{1}{2}+i\tfrac{1}{2}\sqrt{3}), \quad (2k\pi)^{\frac{1}{3}}(\tfrac{1}{2}-i\tfrac{1}{2}\sqrt{3}). \tag{3.28}$$

With a third index running from 1 to 3 to indicate which value of λ_k is being used, we can write:

$$\left.\begin{aligned} \tilde{v}_{1nk} &= -\tilde{w}_{1nk} = \alpha_{nk}[(n/R)\,\tilde{u}'_{1nk}+k\pi\tilde{u}_{1nk}],\\ \tilde{v}_{2nk} &= \tilde{w}_{2nk} = e^{\frac{1}{3}i\pi}\alpha_{nk}[(n/R)\,\tilde{u}'_{2nk}-k\pi\tilde{u}_{2nk}],\\ \tilde{v}_{3nk} &= \tilde{w}_{3nk} = e^{-\frac{1}{3}i\pi}\alpha_{nk}[(n/R)\,\tilde{u}'_{3nk}-k\pi\tilde{u}_{3nk}] \end{aligned}\right\} \tag{3.29}$$

and

$$\left.\begin{aligned} \tilde{v}'_{1nk} &= -\tilde{w}'_{1nk} = -\alpha_{nk}[(n/R)\,\tilde{u}_{1nk}-k\pi\tilde{u}'_{1nk}],\\ \tilde{v}'_{2nk} &= \tilde{w}'_{2nk} = -e^{\frac{1}{3}i\pi}\alpha_{nk}[(n/R)\,\tilde{u}_{2nk}+k\pi\tilde{u}'_{2nk}],\\ \tilde{w}'_{3nk} &= \tilde{v}'_{3nk} = -e^{-\frac{1}{3}i\pi}\alpha_{nk}[(n/R)\,\tilde{u}_{3nk}+k\pi\tilde{u}'_{3nk}], \end{aligned}\right\} \tag{3.30}$$

where

$$\alpha_{nk} = \frac{\beta_k}{n^2R^{-2}+k^2\pi^2} \tag{3.31}$$

and

$$\beta_k = (2kn)^{\frac{1}{3}}. \tag{3.32}$$

We postpone the determination of the $\tilde{u}_{1nk}$'s and $\tilde{u}'_{1nk}$'s until we determine the appropriate boundary conditions, and proceed with the investigation of the $E^{\frac{1}{4}}$ part of the layer. We define the stretched co-ordinate η as follows:

$$\eta = (R-r)\,E^{-\frac{1}{4}}. \tag{3.33}$$

Again omitting some simple calculations, we can see that the boundary-layer fields must be scaled thus:

$$u = E^{a+\frac{1}{4}}\bar{u}, \quad v = E^{a}\bar{v}, \quad w = E^{a+\frac{1}{4}}\bar{w}, \tag{3.34}$$

where $\bar{u}$, $\bar{v}$ and $\bar{w}$ are of $O(1)$ and satisfy the following equations:

$$\bar{w}_z = \tfrac{1}{2}\bar{v}_{\eta\eta\eta}, \quad \bar{v}_z = 0, \quad -\bar{u}_\eta+(1/R)\,\bar{v}_\theta = 0. \tag{3.35}$$

From (3.35), we can immediately see that $\bar{u}$ and $\bar{v}$ are independent of z and that $\bar{w}$ is a linear function of z (property (ii)). The boundary conditions (3.14) imply that

$$\left.\begin{aligned} \bar{w} &= \tfrac{1}{2}\bar{v}_\eta \quad &\text{at}\quad z = 1,\\ \bar{w} &= -\tfrac{1}{2}\bar{v}_\eta \quad &\text{at}\quad z = 0. \end{aligned}\right\} \tag{3.36}$$

On solving for $\bar{u}$, $\bar{v}$ and $\bar{w}$ we get

$$\left.\begin{aligned} \bar{u} &= -\frac{1}{\sqrt{2}R}\frac{dA}{d\theta}\exp(-\sqrt{2}\eta),\\ \bar{v} &= A\exp(-\sqrt{2}\eta),\\ \bar{w} &= -\sqrt{2}A(z-\tfrac{1}{2})\exp(-\sqrt{2}\eta), \end{aligned}\right\} \tag{3.37}$$

where A is an unknown function of θ.

We have deduced all the properties of the interior and side wall boundary layer which were stated at the beginning of this paragraph. Let us now return to the boundary conditions which can indeed be written as in (3.1)–(3.3). The conclusion that $b = -\frac{1}{3}$ and that boundary conditions for the $E^{\frac{1}{3}}$ layer fields are indeed those given in (3.4) is straightforward.

If we assume that $a = -\frac{1}{4}$ and $c > 0$, i.e. that $\mathscr{I}(\theta)$ 'excites' the $E^{\frac{1}{4}}$ layer rather than the interior, then (3.2) requires that $\bar{v}$ vanish on the wall, or, using the expression for $\bar{v}$ given in (3.37), that $A \equiv 0$, which entails that $\bar{u}$, $\bar{v}$ and $\bar{w}$ are identically zero. The same contradiction would be reached if we assume that $a = \frac{1}{4}$ and $c = 0$. The only possibility left would be to take $a = 0$ and $c = 0$, which implies that (3.7) and (3.8) are the correct boundary conditions for the interior and $E^{\frac{1}{4}}$ layer fields. Physically this implies that the horizontal transport occurs *via the interior* and not via the side wall boundary layer. Furthermore, since $c = 0$, the horizontal transport via the Ekman layers is at most of $O(E^{\frac{1}{2}})$.

Having determined a, b and c and the appropriate boundary conditions for the $E^{\frac{1}{4}}$ layer, $E^{\frac{1}{3}}$ layer and interior fields, we can complete the solution. Expanding $\tilde{I}$ in a double Fourier series, viz.

$$\tilde{I}(z,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}[I_{nk}\cos n\theta + I'_{nk}\sin n\theta]\cos k\pi z, \tag{3.38}$$

and using (3.6), (3.29) and (3.30), we can write

$$\tilde{u} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{2}{\sqrt{3}}\exp\left(-\tfrac{1}{2}\beta_k\rho\right)\sin\left(\tfrac{1}{2}\sqrt{3}\beta_k\rho+\tfrac{1}{3}\pi\right)\cos k\pi z\,(I_{nk}\cos n\theta + I'_{nk}\sin n\theta), \tag{3.39}$$

$$\tilde{v} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{2}{\sqrt{3}}\,\alpha_{nk}\exp\left(-\tfrac{1}{2}\beta_k\rho\right)\sin\left(\tfrac{1}{2}\sqrt{3}\beta_k\rho\right)\cos k\pi z$$
$$\times\{([n/R]\,I'_{nk} - k\pi I_{nk})\cos n\theta - ([n/R]\,I_{nk} + k\pi I'_{nk})\sin n\theta\}, \tag{3.40}$$

$$\tilde{w} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{2}{\sqrt{3}}\,\alpha_{nk}\exp\left(-\tfrac{1}{2}\beta_k\rho\right)\sin\left(\tfrac{1}{2}\sqrt{3}\beta_k\rho\right)\sin k\pi z$$
$$\times\{([n/R]\,I'_{nk} - k\pi I_{nk})\cos n\theta - ([n/R]\,I_{nk} + k\pi I'_{nk})\sin n\theta\}. \tag{3.41}$$

It is interesting to note how I_{nk} and I'_{nk} enter into the expressions for $\tilde{v}$ and $\tilde{w}$ and that a θ-symmetry in the injection velocity I does not entail an analogous symmetry of $\tilde{v}$ and $\tilde{w}$.

The boundary-value problem for the interior flow is solved next. Using the method of separation of variables, the solution of (3.17) can be written as

$$P_0(r,\theta) = \sum_{n=1}^{\infty}(r/R)^n\,[A_n\cos n\theta + B_n\sin n\theta] + \text{const.} \tag{3.42}$$

Using the boundary condition (3.7) written in terms of P_0, viz.

$$-\frac{1}{2R}\frac{\partial P_0}{\partial\theta}\bigg|_{r=R} = \mathscr{I}(\theta)$$

we can determine the values of the A_n's and B_n's. Expanding $\mathscr{I}(\theta)$ in Fourier series:

$$\mathscr{I}(\theta) = \sum_{n=1}^{\infty}(\mathscr{I}_n\cos n\theta + \mathscr{I}'_n\sin n\theta),\dagger \tag{3.43}$$

we deduce that

$$P_0(r,\theta) = \sum_{n=1}^{\infty}\frac{2R}{n}\left(\frac{r}{R}\right)^n[\mathscr{I}'_n\cos n\theta - \mathscr{I}_n\sin n\theta]. \tag{3.44}$$

† The summation index n runs from 1 to infinity since $\int_0^{2\pi}\mathscr{I}(\theta)\,d\theta = 0$.

The interior flow is driven by the vertical average of the injection velocity $I(z, \theta)$. As a result of the smearing out of the z-dependence, a point source acts like a vertical line source in so far as the interior fields are concerned.

Finally, using (3.8) we can determine $A(\theta)$ and thus complete the determination of the $E^{\frac{1}{4}}$ layer fields:

$$A(\theta) = -\frac{1}{2}\frac{\partial P_0}{\partial r}\bigg|_{r=R}, \tag{3.45}$$

where V_0 has been written in terms of P_0.

The boundary condition (3.1) for the radial velocity is satisfied to $O(E^{\frac{1}{4}})$. Physically, this is due to the fact that the $E^{\frac{1}{4}}$ layer is divergent and the radial mass flux in and out of this layer induces a secondary interior circulation of order $E^{\frac{1}{4}}$, viz. $(\mathbf{V}_4, P_4)$ which can be shown to be two-dimensional and irrotational. The boundary condition (3.3) for the vertical velocity is also satisfied to $O(E^{\frac{1}{4}})$, $\overline{w}$ being different from zero at the wall. Both because $\overline{w}$ is a function of z and because W_4 is identically zero, we should introduce a correction to $\tilde{w}$ to that order. Proceeding in this manner, we could derive appropriate boundary-value problems for the various terms in a formal $E^{\frac{1}{12}}$ expansion. We shall however not pursue these calculations, which are not very instructive.

4. Discussion

We have seen that when fluid is injected along the vertical wall of a straight cylinder of circular cross-section, a vertical shear layer of thickness $E^{\frac{1}{3}}$, in which the vertical transport occurs, is set up along this wall. The pressure gradient thus induced forces an azimuthal flow of $E^{-\frac{1}{3}}$. However, the z-average of this zonal flow is zero and therefore there is no net horizontal transport via this boundary layer. Rather, the horizontal transport occurs via the interior of the fluid in which the flow is identical to the potential flow due to the z-average of the source–sink distribution $\mathscr{I}(\theta)$ in the case in which the system fluid–cylinder is not rotating. Finally, we have seen that the Ekman layers, which are non-divergent, turned out to play a passive role in the transport process.

The derivation of the above-mentioned results rests on certain properties of both the interior and the side wall boundary layer fields which are quite general, and it is clear that the same conclusions hold for a straight cylinder of arbitrary cross-section (which is simply connected).

At first sight the fact that the horizontal transport takes place via the interior is rather baffling and seems difficult to reconcile with some markedly different results obtained by Hide (1967), Lewellen (1965) and others who also considered flows induced by sources and sinks in rotating fluids. Lewellen found that, when fluid is uniformly injected through a porous cylinder and withdrawn from a second concentric porous cylinder, the radial transport takes place by means of the Ekman layers. This was also the case for the flow due to a vertical line source and a vertical line sink inside a cylinder which Hide has considered. This apparent contradiction can be understood by invoking a theorem derived by Taylor (1917) which points out the analogy between potential flows 'due' to a moving body and the corresponding flows when the system fluid–boundary is rotating. Roughly speaking, this theorem states that the streamfunction for an irrotational flow (with respect to a rotating frame) is identical to that of the corresponding flow for which the system fluid–boundary is at rest, provided that the latter streamfunction can be looked upon as a pressure field. We have already seen that, in a rotating fluid, the pressure is a streamfunction for the interior flow [see (3.13)] and that, whenever the top and bottom Ekman layers are non-divergent, this pressure field is a harmonic function [see (3.17)]. When can we pursue the analogy with potential flows further and say that the proper boundary conditions for this streamfunction are:

$$\partial P/\partial s = -2\mathscr{I}_\ell(s) \quad \text{on} \quad \Gamma_\ell, \tag{4.1}$$

where s is the co-ordinate along the ℓth boundary Γ_ℓ and $\mathscr{I}_\ell$ the injection velocity? Whenever (4.1) turns out to be the proper boundary condition, the interior flows will be identical (when viewed within a frame fixed with respect to the boundary) whether or not the system fluid–boundary is rotating. Since there can be no 'sources' and 'sinks' of pressure, $\oint dP$ must always vanish, and it is clear that (4.1) cannot be a meaningful boundary condition unless

$$\oint_{\Gamma_\ell} \mathscr{I}_\ell(s)\, ds = 0. \tag{4.2}$$

This condition is violated in the problems considered by Hide and Lewellen. In fact, since P is a streamfunction for the interior motion, and since

$$\oint_{\Gamma} dP$$

is also zero for a curve Γ which encloses a source, we immediately conclude that there can be no transport from a 'monopole' source (or sink) via the interior, and as a result a streamline must wind around a source rather than issue from it. The horizontal transport from a monopole source to a monopole sink must take place through regions where the flow is rotational and/or not two-dimensional, such as Ekman layers. In the present problem, the condition (4.2) is satisfied [see (2.9)], and therefore the horizontal transport can and indeed does occur via the interior.

If we generalize the present problem by relaxing the requirement that there can be no net flux across a vertical boundary,† the horizontal transport will occur partly via the interior and partly via the Ekman layers. This can be seen by breaking up the injection velocity $I_\ell(z,s)$ along the ℓth boundary as follows:

$$I_\ell(z,s) = \hat{I}_\ell + \mathscr{I}_\ell(s) + \tilde{I}_\ell(z,s), \tag{4.3}$$

where

$$\hat{I}_\ell = \oint_{\Gamma_\ell} ds \int_0^1 dz\, I_\ell(z,s), \tag{4.4}$$

i.e. the monopole component of the distribution I_ℓ, and

$$\hat{I}_\ell + \mathscr{I}_\ell(s) = \int_0^1 dz\, I_\ell(z,s). \tag{4.5}$$

† The cross-section of the cylindrical container must consequently be multiply connected.

The horizontal transport due to the $\hat{I}_\ell$'s takes place by means of the Ekman layers whereas that due to the $\mathscr{I}_\ell$'s occurs through the interior.

A simple experiment was performed in an attempt to visualize both the interior and Ekman layer flows. The apparatus is shown in figure 1, plate 1 and consists of a straight circular cylinder with two holes on the vertical side wall through which fluid is injected and withdrawn. The flow rate and angular rotation were such that both the Rossby and Ekman numbers were of the order of 10^{-4} for all the experiments, thus ensuring the validity of both the linearization and the boundary layer approximation. The experiments were not entirely reliable because of extraneous oscillations in the flow due to the slow dripping of the fluid through the exhaust pipe and because the pipe at the centre of the upper surface acted as an additional weak source. However, in spite of these inadequacies, dye introduced at the top of the cylinder clearly showed the existence of interior motions in the direction of the sink (see, figure 2, plate 2). Furthermore, drops of dye which touched the bottom and entered the Ekman layer failed to indicate the existence of large velocities which should have been observed if the transport was taking place via the Ekman layer.

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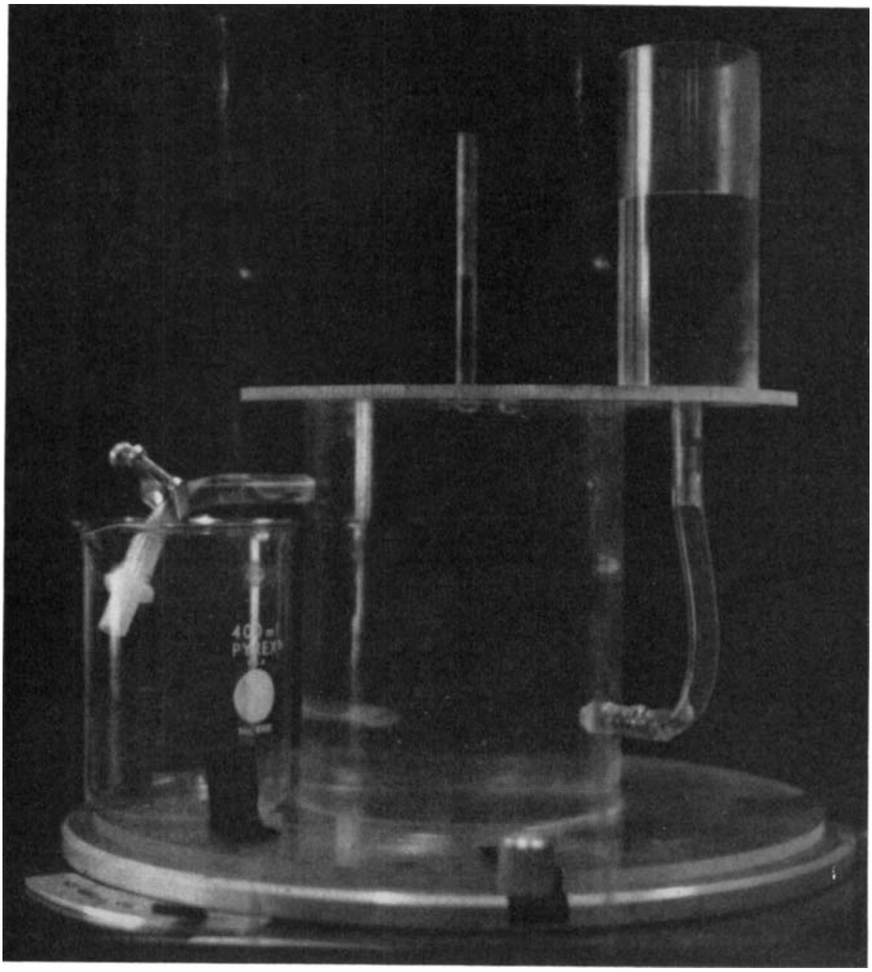

FIGURE 1. Picture of the apparatus. An arbitrary distribution of sources and sinks can be thought of as being made up by pairs of point sources and sinks of equal strength.

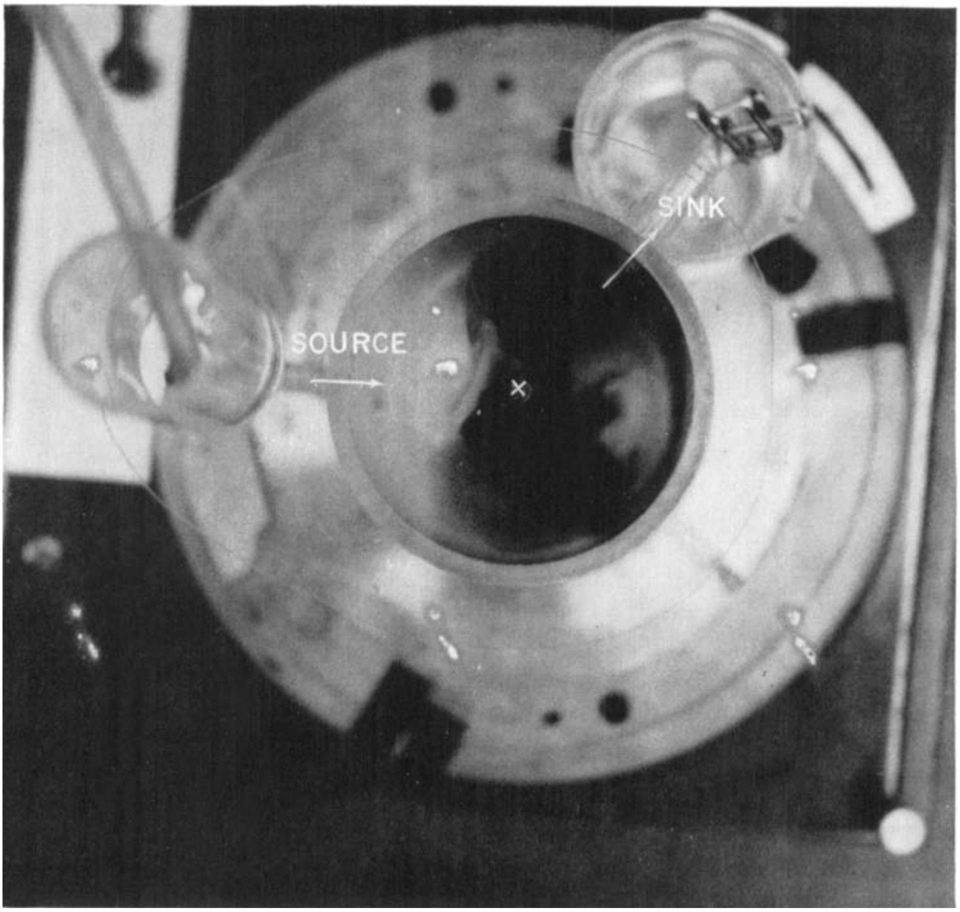

FIGURE 2. Dye injected at the centre of the cylinder has moved toward the sink, thus displaying the existence of interior motions with source-sink streamlines.